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The 0^+ predominance in nuclear physics: single-j shell study

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Abstract

The probability of a state with spin I to be the ground state in many-body systems is studied. Single-j shells with four-particle systems are examined in detail. It is shown that the structure coefficients give a clue to understand the problem why the spin-0 state is most likely to be the ground state.

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1. Introduction

Order and chaos are usually completely conflicting concepts. However, recently Johnson *et al* discovered a very interesting phenomenon [1]: the predominance of 0^+ ground state (0g.s.) of even fermion systems is obtained by diagonalizing a two-body scalar Hamiltonian, the strength of which is randomly determined. This means that the orderly spectra appear as a consequence of chaotic interactions. In nuclear physics, the angular momenta of the ground states of even–even nuclei are always 0^+ without an exception, which is believed to be a consequence of the strong attractive short range interaction (pairing interaction) between like nucleons. However, the work by Johnson *et al* suggests that the 0g.s. predominance arises from the intrinsic features of the model space, and is independent of the specific character of the nuclear force. Already many works have been done along the lines of this discovery [2–13], and one of the most interesting and challenging aspects of this problem is how to understand this observation.

There have been many efforts to understand the 0g.s. predominance. It was shown [8] first in the sd-shell that this phenomenon is not an outcome of time reversal symmetry. Instead, it might be rather a reflection of a large distribution width of the states. In [9], Mulhall *et al* discussed the 0g.s. predominance within single-j shells by using geometric chaoticity and uniformly changed random interactions. However, the general behaviour of the predicted 0g.s. probabilities [9] is quite different from those obtained by diagonalizing the Hamiltonian using random interactions. To simplify the problem, sp- and sd-bosons are extensively studied by Bijker, Frank and Kusnezov [10, 11]. In single-j shells, it was shown that the width is not the answer to the 0g.s. predominance [12]. Instead a feature of the structure coefficients was suggested to provide a reasonable explanation of the angular momentum distribution of the 0g.s. It was assumed [12], however, that the off-diagonal matrix elements were neglected as an approximation. Because of this approximation, the method proposed therein has the disadvantage that it is inapplicable to more complicated cases, e.g. large single-j shells.

It is easily noted that the disadvantage in [12] will be removed if we discuss the Ig.s. probability for the mean energy of each angular momentum I. It is shown that the mean energies of angular momentum I states are linear combinations of two-body matrix elements. In such cases one could use the integral formula given in [12, 14] to predict the 0g.s. probabilities for the mean energy of $I = 0$ states. Below we shall focus on single-j shells, but it is emphasized that the discussion in this paper is valid in general cases, such as nucleons in many-j shells, sd-boson systems, etc.

In this paper we study the probability of a state with angular momentum I to be the ground state in many-body systems when two-body interactions are completely random. Single-j shells with four-particle systems are extensively examined. It is shown that the feature of the structure coefficients gives a clue to understand the problem why the spin $I = 0$ state is most likely to be the ground state. This argument can be generalized to many-j systems and boson systems.

2. Phenomenology

We define the terminology Ig.s. as follows:

Ig.s. $=$ probability of I to be the angular momentum of the ground state.

Then the rules we obey are described as follows for any many-body system:

- the Hamiltonian consists of scalar two-body interactions with all possible combinations;
- two-body interactions are randomly taken with normal (Gaussian) distribution centred on zero.

In this paper we deal with only single-j shells; thus we specify the interactions more specifically. First j denotes the value of angular momentum of a single-j shell and n, the number of particles in the shell. Two-body interactions are expressed as follows:

$$H = \sum_{J=0}^{2j-1} \sqrt{2J+1} G_J \left[A^{\dagger(J)} \tilde{A}^{(J)} \right]^{(0)} \tag{1}$$

with

$$A^{\dagger(J)} = \frac{1}{\sqrt{2}} [a_j^\dagger a_j^\dagger]^{(J)} \qquad \tilde{A}^{(J)} = -\frac{1}{\sqrt{2}} [\tilde{a}_j \tilde{a}_j]^{(J)} \qquad (J = 0, 2, \ldots, 2j-1). \tag{2}$$

Here the strength of the interactions (G_J) is determined according to normal distribution of variance $\sigma^2 = 1$ (Gaussian distribution), that is,

$$P(G_J) = \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{(G_J)^2}{2} \right]. \tag{3}$$

The matrix elements of the two-body interactions are calculated as follows:

$$H_{I\beta'\beta} \equiv \langle j^n I\beta' | H | j^n I\beta \rangle = \sum_J \alpha^J_{I\beta'\beta} G_J \tag{4}$$

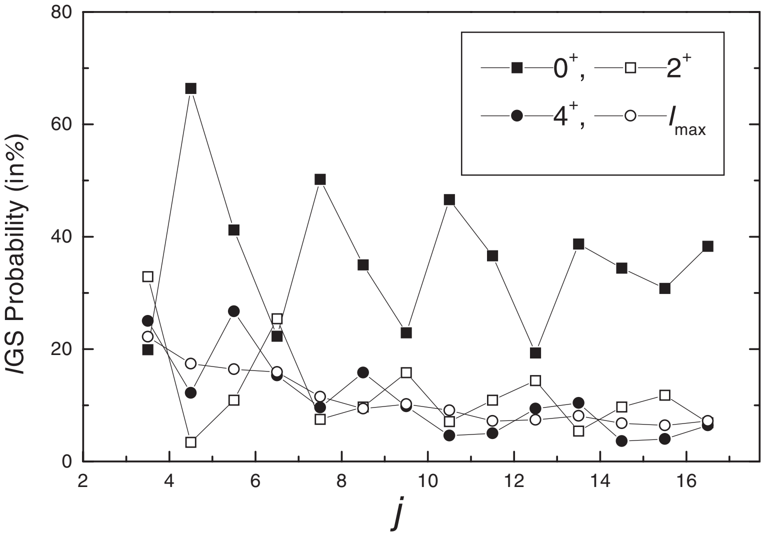

Figure 1. The Ig.s. probability (large probability of finding I to be the angular momentum of the ground state) for four-particle systems with different j.

where β is the additional quantum number which specifies the states uniquely and the structure coefficients $\alpha^J_{I\beta'\beta}$ are defined by

$$\begin{aligned}\alpha^J_{I\beta'\beta} &\equiv \langle j^n I\beta'|\sqrt{2J+1}\left[A^{\dagger(J)}\tilde{A}^{(J)}\right]^{(0)}|j^n I\beta\rangle \\ &= \frac{n(n-1)}{2}\sum_{K,\gamma}\langle j^{n-2}K\gamma, j^2J|\}j^nI\beta'\rangle\langle j^{n-2}K\gamma, j^2J|\}j^nI\beta\rangle. \end{aligned} \tag{5}$$

Here $\langle j^{n-2}K\gamma, j^2J|\}j^nI\beta'\rangle$ are the two-body coefficients of fractional parentage (cfps). Using the orthonormal relation for cfps we have a sum rule for $\alpha^J_{I\beta'\beta}$, that is,

$$\sum_J \alpha^J_{I\beta'\beta} = \frac{n(n-1)}{2}\delta_{\beta'\beta}. \tag{6}$$

Note that this is not the only sum rule for $\alpha^J_{I\beta'\beta}$. For instance, we have another sum rule,

$$\sum_J J(J+1)\alpha^J_{I\beta\beta'} = [I(I+1) + j(j+1)n(n-2)]\,\delta_{\beta\beta'}. \tag{7}$$

In figure 1 we show examples of the 0g.s. predominance. We assume the Box–Muller method to produce random Gaussian two-body interactions centred on zero. We produce 1000 sets of G_J according to (3) and diagonalize the Hamiltonian (1). In these calculations j runs from 7/2 to 33/2. The predominance of 0^+ states as ground states is confirmed in this figure. More precisely, for j larger than 15/2, the probability for 0^+ states to be the ground states is confirmed to be always the largest one.

We observe interesting oscillations in figure 1. The maxima of 0g.s. probabilities occur at $j = 9/2$, $j = 15/2$, $j = 21/5$, $j = 27/2$ and so on. Here the shell-model dimension of $I = 0$ states increases by one compared to the previous j value. Thus the relative importance of $I = 0$ states increases at these j points.

2.1. Simple case

In order to get an idea how Ig.s. is determined, we take up a very simple example as an illustration [12]. We take up the example with $j = 7/2$ and $n = 4$. In this case we have the total angular momentum $I = 0, 2^2, 4^2, 5, 6, 8$. The energy matrix elements are expressed explicitly as follows:

$$E_I^{(\beta)} = \langle j^n I\beta|H|j^n I\beta\rangle = \sum_J \alpha^J_{I\beta\beta} G_J. \tag{8}$$

Here energies are calculated by taking only diagonal matrix elements because off-diagonal elements are zero in this special case ($j = 7/2$):

$$\begin{aligned}
E_0 &= \tfrac{\mathbf{3}}{\mathbf{2}}G_0 + \tfrac{5}{6}G_2 + \tfrac{3}{2}G_4 + \tfrac{13}{6}G_6 && 18.2\% \\
E_2^{(1)} &= \tfrac{1}{2}G_0 + \tfrac{11}{6}G_2 + \tfrac{3}{2}G_4 + \tfrac{13}{6}G_6 && 0.9\% \\
E_2^{(2)} &= 0G_0 + G_2 + \tfrac{\mathbf{42}}{\mathbf{11}}G_4 + \tfrac{13}{11}G_6 && 33.3\% \\
E_4^{(1)} &= \tfrac{1}{2}G_0 + \tfrac{5}{6}G_2 + \tfrac{5}{2}G_4 + \tfrac{13}{6}G_6 && 0.0\% \\
E_4^{(2)} &= 0G_0 + \tfrac{\mathbf{7}}{\mathbf{3}}G_2 + G_4 + \tfrac{8}{3}G_6 && 23.0\% \\
E_5 &= 0G_0 + \tfrac{8}{7}G_2 + \tfrac{192}{77}G_4 + \tfrac{26}{11}G_6 && 0.0\% \\
E_6 &= \tfrac{1}{2}G_0 + \tfrac{5}{6}G_2 + \tfrac{3}{2}G_4 + \tfrac{19}{6}G_6 && 0.0\% \\
E_8 &= 0G_0 + \tfrac{10}{21}G_2 + \tfrac{129}{77}G_4 + \tfrac{\mathbf{127}}{\mathbf{33}}G_6 && 24.2\%
\end{aligned}$$

where the last number indicates the numerical probabilities of Ig.s., using the following integration formula. Here the largest $\alpha^J_{I\beta\beta}$ among all α for a given J is specified in bold font. In the above equations one can see that there is a good correspondence between the largest $\alpha^J_{I\beta\beta}$ for a given J and the large Ig.s. It should be noted that this example does not show the 0g.s. predominance, though the 0g.s. probability is larger than the geometric value (12.5%). This is because the value of j is small. As in figure 1, the 0g.s. predominance is confirmed if $j \geqslant 13/2$.

A simple qualitative argument can be given as follows. For simplicity let us assume that there is only one state for each angular momentum I,

$$E_I = \sum_J \alpha^J_I G_J. \tag{9}$$

Here we have used the abbreviation α^J_I which stands for diagonal $\alpha^J_{I\beta\beta}$. Suppose $\alpha^{J'}_{I'}$ is the largest for a fixed J' and $G_{J'} < 0$. Then

$$\begin{aligned}
\Delta E_I &\equiv E_I - E_{I'} \\
&= \left(\alpha^{J'}_I - \alpha^{J'}_{I'}\right)G_{J'} + \sum_{J\neq J'}\left(\alpha^J_I - \alpha^J_{I'}\right)G_J \\
&\equiv E_I^{(s)} + E_I^{(r)}
\end{aligned} \tag{10}$$

with

$$E_I^{(s)} = \begin{cases} 0 & \text{for } I = I' \\ >0 & \text{others.} \end{cases} \tag{11}$$

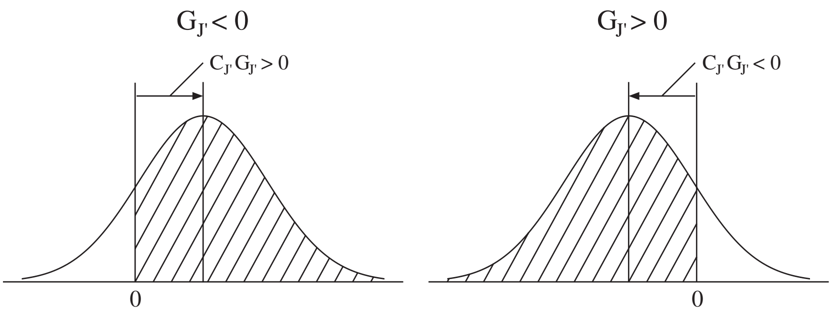

Figure 2. The probability distribution of ΔE_I. Here $C_{J'} = \alpha_I^{J'} - \alpha_{I'}^{J'} < 0$. The centre of the distribution is shifted by $E_I^{(s)}$.

Here $E_I^{(s)}$ gives the shift of the centre of the distribution and $E_I^{(r)}$ are randomly distributed by $G_{J \neq J'}$. Figure 2 explains why ΔE_I has a large probability (shown by the shadow) to be positive when $G_{J'} < 0$ and vice versa.

In this simple case we have the analytical expression for the 0g.s. expressed in an integration form as

$$0\text{g.s.} = \int \mathrm{d}G_0 P(G_0) \int \mathrm{d}G_2 P(G_2) \int \mathrm{d}G_4 P(G_4) \int \mathrm{d}G_6 P(G_6)$$
$$\times \int \mathrm{d}E_0 \int_{E_0} \mathrm{d}E_2^{(1)} \int_{E_0} \mathrm{d}E_2^{(2)} \cdots \int_{E_0} \mathrm{d}E_8\, \delta\left(E_0 - \sum_J \alpha_0^J G_J\right)$$
$$\times \delta\left(E_2^{(1)} - \sum_J \alpha_{2^{(1)}}^J G_J\right) \cdots \delta\left(E_8 - \sum_J \alpha_8^J G_J\right). \tag{12}$$

This expression is exact because we do not need to take into account off-diagonal elements, and henceforth, configuration mixings. This expression is valid even if we replace $P(G_J)$ by other distributions, such as a uniform distribution. In a similar way other Ig.s. probabilities can be evaluated using integration formulae. We have evaluated equation (12) numerically and confirmed that the numerical Ig.s. probabilities are reproduced exactly within numerical error bars.

2.2. More complicated systems

In more complicated systems we have many levels for each I. Thus we cannot ignore the configuration mixings. Since each individual energy level is considered to be a deviation from the mean energy level for the angular momentum I, we consider the mean energy level. The mean energy level for I is expressed in terms of G_J as follows:

$$\bar{E}_I \equiv \frac{1}{n_\beta} \sum_{\beta=1}^{n_\beta} E_{I\beta} = \frac{1}{n_\beta} \mathrm{Tr}\,(H)_I = \frac{1}{n_\beta} \sum_{\beta=1}^{n_\beta} \sum_J \alpha_{I\beta\beta}^J G_J = \sum_J \bar{\alpha}_I^J G_J \tag{13}$$

with $\bar{\alpha}_I^J = \frac{1}{n_\beta} \sum_{\beta=1}^{n_\beta} \alpha_{I\beta\beta}^J$. Here $E_{I\beta}$ is an eigenenergy of the system with configuration mixings and the number of levels belonging to angular momentum I is denoted by n_β.

We have the following notable advantages to consider the mean energy level:

- We have only one state for each I and we can use the argument of the previous section. Equation (9) is replaced by
$$E_I = \sum_J \alpha_I^J G_J \Rightarrow \bar{E}_I = \sum_J \bar{\alpha}_I^J G_J. \tag{14}$$
In this way we can employ the integration formula (12) in the same way.
- Since the trace is basis independent, $\bar{\alpha}_I^J$ is basis independent.
- We can use the sum rule for the mean $\bar{\alpha}_I^J$,
$$\sum_J \bar{\alpha}_I^J = \frac{n(n-1)}{2}. \tag{15}$$
- The generalization to many-j shells is easy.

In order to address the original problem, we should consider each level. We consider the variance of energy levels for each angular momentum I. The variance of energy levels is calculated as follows. First the mean energy and mean energy variance are expressed in terms of Hamiltonian matrix elements,

$$(\bar{E}_I)^2 = \left(\frac{1}{n_\beta}\sum_{\gamma=1}^{n_\beta} E_{I\gamma}\right)^2 = \frac{1}{n_\beta^2}\left(\sum_{\beta=1}^{n_\beta}(H_{I\beta\beta})^2 + 2\sum_{\beta'>\beta=1}^{n_\beta} H_{I\beta'\beta'}H_{I\beta\beta}\right) \tag{16}$$

$$\overline{E_I^2} = \frac{1}{n_\beta}\sum_{\gamma=1}^{n_\beta} E_{I\gamma}^2 = \frac{2}{n_\beta}\sum_{\beta'>\beta=1}^{n_\beta}(H_{I\beta'\beta})^2 + \frac{1}{n_\beta}\sum_{\beta=1}^{n_\beta}(H_{I\beta\beta})^2. \tag{17}$$

Then taking the ensemble average (taking G_J random), we have

$$\begin{aligned}\left\langle(\bar{E}_I)^2\right\rangle &= \left\langle \frac{1}{n_\beta^2}\left(\sum_{\beta=1}^{n_\beta}(H_{I\beta\beta})^2 + 2\sum_{\beta'>\beta=1}^{N} H_{I\beta'\beta'}H_{I\beta\beta}\right)\right\rangle \\ &= \frac{1}{n_\beta^2}\sum_{J=0}^{2j-1}\left(\sum_{\beta=1}^{n_\beta}\alpha_{I\beta\beta}^J\right)^2 = \sum_{J=0}^{2j-1}\left(\bar{\alpha}_I^J\right)^2 \end{aligned}\tag{18}$$

$$\begin{aligned}\left\langle\overline{E_I^2}\right\rangle &= \left\langle \frac{2}{n_\beta}\sum_{\beta'>\beta=1}^{n_\beta}(H_{I\beta'\beta})^2 + \frac{1}{n_\beta}\sum_{\beta=1}^{n_\beta}(H_{I\beta\beta})^2\right\rangle \\ &= \frac{1}{n_\beta}\sum_{J=0}^{2j-1}\sum_{\beta',\beta=1}^{n_\beta}\left(\alpha_{I\beta'\beta}^J\right)^2. \end{aligned}\tag{19}$$

Therefore the variance for each level is calculated as follows. First the variance of a certain individual level $E_{I\beta}$ is given by

$$(\sigma_{I\beta})^2 = \left\langle(E_{I\beta} - \bar{E}_I)^2\right\rangle. \tag{20}$$

Since each individual level is equally independent, we have the common variance,

$$\begin{aligned}(\sigma_I)^2 &= \overline{(\sigma_{I\beta})^2} = \overline{\left\langle(E_{I\beta} - \bar{E}_I)^2\right\rangle} \\ &= \left\langle\overline{E_I^2} - (\bar{E}_I)^2\right\rangle = \frac{1}{n_\beta}\sum_{J=0}^{2j-1}\sum_{\beta',\beta=1}^{n_\beta}\left(\alpha_{I\beta'\beta}^J\right)^2 - \sum_{J=0}^{2j-1}\left(\bar{\alpha}_I^J\right)^2. \end{aligned}\tag{21}$$

Table 1. The Ig.s. probability (up to fifth largest case) and Ig.s. average probability for $j = 31/2$ with four particles. The Ig.s. average probabilities are calculated in terms of the trace of the Hamiltonian H for each angular momentum I. The SM-d is the shell model dimension for each I. The definitions of σ_I, γ_I, σ_I^T, $\tilde{\sigma}_I$, $\tilde{\sigma}_I^T$ and $\Delta\bar{E}_I$ are given in the text.

I	Ig.s. probability (%)	Ig.s. average probability (%)	SM-d	σ_I	γ_I	σ_I^T	$\tilde{\sigma}_I$	$\tilde{\sigma}_I^T$	$\Delta\bar{E}_I$
0	30.8	11.5	5	3.15	1.57	3.52	2.14	2.65	−3.67
2	11.8	3.7	10	2.76	1.54	3.16	1.61	2.23	−4.26
4	4.0	0.4	14	2.54	1.55	2.97	1.39	2.08	−4.32
6	7.6	1.1	17	2.44	1.57	2.90	1.30	2.04	−4.38
56	6.4	23.3	1	0.00	4.10	4.10	0.00	4.10	0.00

We call $(\sigma_I)^2$ the individual width for I. The above expression is simplified as

$$(\sigma_I)^2 = \frac{1}{n_\beta}\sum_{J=0}^{2j-1} \mathrm{Tr}\left[\left(\alpha_I^J - \bar{\alpha}_I^J\right)^2\right]. \tag{22}$$

Here it is seen that the deviation from the mean value is important to get a larger width.

In table 1 we show the Ig.s. probability (up to fifth largest case) and Ig.s. average probability for the average energies in the case of $j = 31/2$ and $n = 4$. For the average energies the largest probability to be the ground state is found for the highest spin $I = I_{\max} = 56^+$. The probability of the 0^+ average energy to be the lowest one comes next. However, we should be aware that there is only one state for $I = 56^+$. On the other hand, there are five 0^+ states with the largest individual width σ_I. Some of them are pushed down far from their average energy. Thus we can expect that the probability of a 0^+ state to be the ground state is much larger than the 0g.s. average value (11.5%). In fact that happens in this example.

2.3. Generalization of the integration formula

Since the distribution for the mean energy of the angular momentum I is expressed as

$$P(\bar{E}_I) = \prod_{J=0}^{2j-1} \frac{1}{\sqrt{2\pi}} \int \mathrm{d}G_J \exp\left[-\frac{(G_J)^2}{2}\right] \delta\left(\bar{E}_I - \sum_{J=0}^{2j-1} \bar{\alpha}_I^J G_J\right) \tag{23}$$

it is simplified as

$$P(\bar{E}_I) = \frac{1}{\sqrt{2\pi}\gamma_I} \exp\left[-\frac{(\bar{E}_I)^2}{2\,(\gamma_I)^2}\right] \tag{24}$$

with $(\gamma_I)^2 = \sum_{J=0}^{2j-1}\left(\bar{\alpha}_I^J\right)^2$. We call $(\gamma_I)^2$ the width of mean energy levels for I. Then the total width of each level is given by

$$\left(\sigma_I^T\right)^2 = (\sigma_I)^2 + (\gamma_I)^2 = \frac{1}{n_\beta}\sum_{J=0}^{2j-1} \mathrm{Tr}\left[\left(\alpha_I^J\right)^2\right]. \tag{25}$$

We should not simply use this total width for arguing Ig.s. From numerical studies we know that each level distributes according to a normal distribution. Therefore, it seems plausible to consider the total width of the Gaussian distribution of each I for an account of Ig.s., i.e. if the total width of levels of a certain I is the largest among all, the argument that the Ig.s. probability is large is easy to take for granted. This argument is not true. From numerical studies we observe that the state with highest angular momentum $I = I_{\max}$ gives the largest total width and that with $I = 0$ gives the second largest. This happens due to the strong correlation between each state because we use a two-body random ensemble (TBRE) and not a Gaussian orthogonal ensemble (GOE). This can be easily seen from the simple integral formula (12). Here G_J are the independent parameters and $E_I^{(\beta)}$ are not. In general we cannot express this kind of integration formula explicitly because the diagonalization procedure prohibits us from expressing each $E_{I\beta}$ in terms of G_J due to mixing. Here we make the assumption that the correlation between $E_{I\beta}$ is wholly absorbed in the mean energy levels and that an individual level distributes according to the Gaussian distribution of the mean energy level as its centre.

Upon the above assumption, the probability that the energy of a level with I_1 is lower than that of I_2 is given by

$$P\left(E_{I_1} < E_{I_2}\right) = \prod_{J=0}^{2j-1} \int \mathrm{d}G_J \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{(G_J)^2}{2}\right] \int \mathrm{d}\bar{E}_{I_1} \int \mathrm{d}\bar{E}_{I_2}\, \delta\left(\bar{E}_{I_1} - \sum_{J=0}^{2j-1} \bar{\alpha}_{I_1}^J G_J\right)$$
$$\times\, \delta\left(\bar{E}_{I_2} - \sum_{J=0}^{2j-1} \bar{\alpha}_{I_2}^J G_J\right) \int \mathrm{d}E_{I_1} P\left(E_{I_1}, \bar{E}_{I_1}, \sigma_{I_1}\right) \int_{E_{I_1}} \mathrm{d}E_{I_2} P\left(E_{I_2}, \bar{E}_{I_2}, \sigma_{I_2}\right). \tag{26}$$

Here the distribution $P(E_I, \bar{E}_I, \sigma_I)$ is defined as

$$P(E_I, \bar{E}_I, \sigma_I) = \frac{1}{\sqrt{2\pi}\sigma_I} \exp\left[-\frac{(E_I - \bar{E}_I)^2}{2(\sigma_I)^2}\right]. \tag{27}$$

In this derivation we have not made use of the fact that the selected state with E_I should be the lowest state among the states belonging to the same angular momentum I. Suppose that the number of the shell-model dimension belonging to the angular momentum I is n. Then the probability that the lowest state has the energy $E_I^{(1)}$ may be written as

$$\tilde{P}\left(E_I^{(1)}, \bar{E}_I, \sigma_I\right) \propto P\left(E_I^{(1)}, \bar{E}_I, \sigma_I\right) \int_{E_I^{(1)}} \mathrm{d}E_I^{(2)} P\left(E_I^{(2)}, \bar{E}_I, \sigma_I\right)$$
$$\times \cdots \times \int_{E_I^{(1)}} \mathrm{d}E_I^{(n)} P\left(E_I^{(n)}, \bar{E}_I, \sigma_I\right). \tag{28}$$

Equation (28) is difficult to evaluate analytically, so we make another assumption that it is approximately expressed as a single Gaussian distribution:

$$\tilde{P}\left(E_I^{(1)}, \bar{E}_I + \Delta\bar{E}_I, \tilde{\sigma}_I\right) = \frac{1}{\sqrt{2\pi}\tilde{\sigma}_I} \exp\left[-\frac{\left(E_I^{(1)} - \bar{E}_I - \Delta\bar{E}_I\right)^2}{2(\tilde{\sigma}_I)^2}\right]. \tag{29}$$

Here the new width $(\tilde{\sigma}_I)^2$ and the shift from the mean energy $\Delta\bar{E}_I$ are evaluated numerically. It should be noted that these are strongly dependent on the number of the shell-model dimension for the angular momentum I. Finally the probability that the lowest state with angular momentum I_1 is lower than that of I_2 is given approximately as

$$P\left(E_{I_1} < E_{I_2}\right) = \prod_{J=0}^{2j-1} \int \mathrm{d}G_J \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{(G_J)^2}{2}\right] \int \mathrm{d}\bar{E}_{I_1} \int \mathrm{d}\bar{E}_{I_2} \delta\left(\bar{E}_{I_1} - \sum_{J=0}^{2j-1} \bar{\alpha}_{I_1}^J G_J\right)$$
$$\times\, \delta\left(\bar{E}_{I_2} - \sum_{J=0}^{2j-1} \bar{\alpha}_{I_2}^J G_J\right) \int \mathrm{d}E_{I_1} \tilde{P}\left(E_{I_1}, \bar{E}_{I_1} + \Delta\bar{E}_{I_1}, \tilde{\sigma}_{I_1}\right)$$
$$\times \int_{E_{I_1}} \mathrm{d}E_{I_2} \tilde{P}\left(E_{I_2}, \bar{E}_{I_2} + \Delta\bar{E}_{I_2}, \tilde{\sigma}_{I_2}\right). \tag{30}$$

This expression is easily generalized for the 0g.s. as in equation (12).

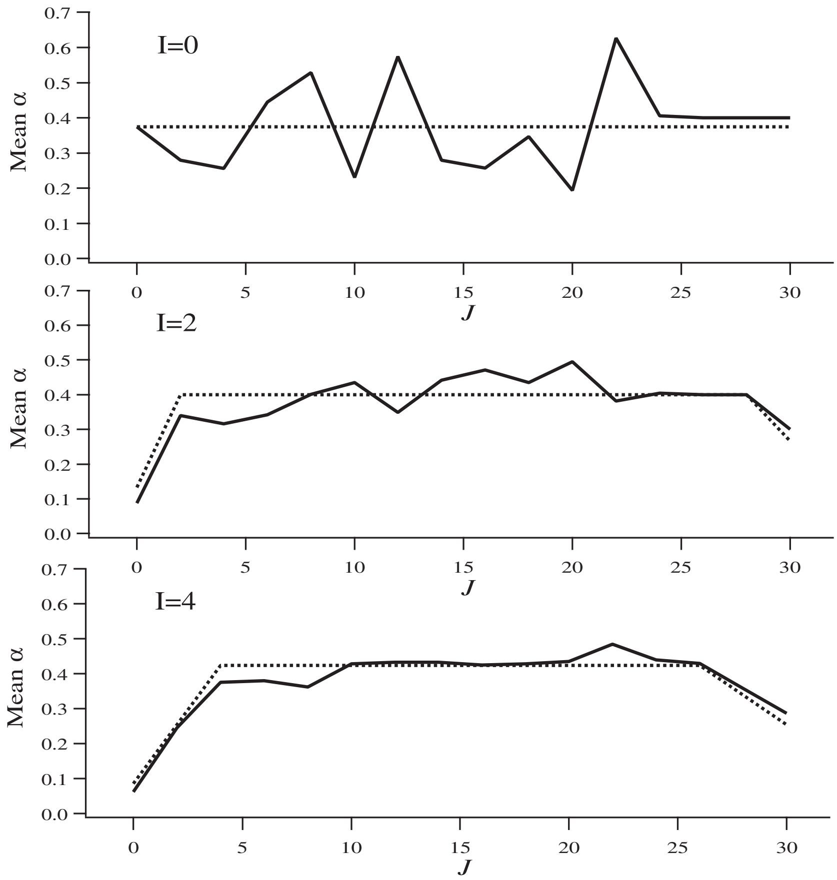

Figure 3. Predicted $\bar{\alpha}_I^J$ (dotted line) and actual values (solid line) for $I = 0, 2, 4$ as a function of J with $j = 31/2$ and $n = 4$.

In table 1 numerically evaluated values of $\tilde{\sigma}_I$, $\Delta\bar{E}_I$ are given for each angular momentum I. By consulting this table, we can qualitatively understand the 0g.s. predominance. The total width $\tilde{\sigma}_I^T$ is largest for $I = 56$ and the Ig.s. average probability becomes the largest. However, the energy shift $\Delta\bar{E}_I$ is zero because there is only one state for $I = 56$. In contrast, the total width $\tilde{\sigma}_I^T$ is second largest for $I = 0$, but the energy shift is relatively large (-3.67).

3. Microscopic understanding

As we have seen in the previous sections, the 0g.s. are completely determined from the structure of $\alpha_{I\beta\beta}^J$. In this sense the 0g.s. predominance problem is restated as follows. The basis independent structure of $\alpha_{0\beta\beta'}^J$ is in some sense different from the other $\alpha_{I\neq 0\beta\beta'}^J$. In this sense we cannot go further, since the structure of $\alpha_{I\beta\beta}^J$ is determined when we set up a physical model. Each model defines the structure of $\alpha_{I\beta\beta}^J$. Then the next question is what kind of model gives the 0g.s predominance. As far as we know all the nuclear shell models for both fermions and bosons generically give the 0g.s predominance except for the pure d-boson model [15]. However, we can look at this problem in a different way. We may assume that these geometrical factors are determined randomly. Here we take mean α for simplicity,

$$\bar{\alpha}_I^J \equiv \frac{1}{n_\beta}\sum_{\beta=1}^{n_\beta}\left[\frac{n(n-1)}{2}\sum_{K\gamma}(\langle j^{n-2}K\gamma, j^2J|\}j^nI\beta\rangle)^2\right]$$
$$\to \frac{6}{n_\beta}\sum_{\beta=1}^{n_\beta}\sum_{K\gamma}(\langle j^2K\gamma, j^2J|\}j^4I\beta\rangle)^2 \qquad \text{for} \quad n=4. \tag{31}$$

We can assume that each $(\langle j^2K\gamma, j^2J|\}j^4I\beta\rangle)^2$ has the same value. Since the sum rule (12) can determine that value, each mean value of α is determined. For instance, $\bar{\alpha}_{I=0}^J$ for $n=4$ is determined as follows. Since $I=0$, we have $K=J$. It means that we have only one cfp $\langle j^2J, j^2J|\}j^4I=0\rangle$, which eventually determines $\bar{\alpha}_{I=0}^J = \frac{6}{j+1/2} = 0.375$ for $j=31/2$.

In figure 3 we show predicted $\bar{\alpha}_I^J$ (dotted line) and actual values (solid line) for $I=0,2,4$ as a function of J. We note that the predicted values well reproduce those of actual $\bar{\alpha}_I^J$, but fluctuations (differences between predicted and actual ones) are large for $I=0$ compared to other I. Since the crude value of $\bar{\alpha}_I^J$ is similar for all I, it is likely that $I=0$ state has the largest $\bar{\alpha}_I^J$ for a given J because of fluctuations. Our numerical calculation suggests that this fluctuation is related to the number of possible cfp's, that is, the greater the number of contributed cfp's, the less the fluctuation. However, we need a further study along this line. So far this statistical theory is only successful for the mean energy level. We are searching for a theory applicable to each individual level [16].

4. Summary and conclusions

In this paper we have shown our results for single-j shells with $n=4$. The results here will be generalized to many-j shells, boson systems and so on. A strong predominance of the spin-0 ground states is confirmed for systems with simple configurations for $j=7/2$–$33/2$. The probability of spin-0 ground states is stably the largest for 4-fermion systems with $j \geqslant 15/2$. For the mean energy level we have shown that the spin-0 predominance is largely attributed to the presence of large fluctuations in mean values of structure coefficients α. We can explain why there are large fluctuations in α for spin-0 states by assuming that cfp's are randomly determined and the fact that the number of contributed cfp's is small for $I=0$. We have thus found an explanation of the large probability of the 0^+ ground state for the mean energy.

There remains, however, a question why we have the predominance of the 0^+ ground state in the individual levels. For that purpose our integration formula has been extended for this individual case as in equation (30). Although this expression can be numerically evaluated in principle, it is too complicated to evaluate for large j values. Moreover we still do not understand the physics behind this problem even after evaluating this formula. However, the qualitative understanding is obtained by looking at the new width $(\tilde{\sigma}_I)^2$ and the shift from the mean energy $\Delta\bar{E}_I$.

Recently another efficient phenomenological approach is invented [16]. We first set one of the two-body matrix elements of the problem to -1 and all the rest to zero and then see which angular momentum I gives the lowest eigenvalue among all of the eigenvalues of this many-body system. If the number of independent two-body interaction matrix elements is N, the above procedure is repeated N times, with each of the matrix elements assuming the privileged role of being set to -1. After all N calculations have been done, we simply count how many times (denoted as $\mathcal{N}_I$) the angular momentum I gives the lowest eigenvalue. Finally, the Ig.s. probability is predicted as $\mathcal{N}_I/N$.

The above procedure is confirmed to give a very good approximation for the original Ig.s. probabilities [16]. We understand that this is not just a coincidence. However, the above procedure is still a phenomenology and the fundamental physics behind this should be sought. This is still an open question.

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